Indian Statistical Institute Admission Test for B.Math/B.Stat 2016 - Solved Paper

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Q1

In a sports tournament of n players, each pair of players plays exactly one match against each other. There are no draws. Prove that the players can be arranged in an order $P_1, P_2, ..., P_n$, such that P_i defeats P_{i+1} $\forall$ $i = 1, 2, ..., n-1$.

Solution

We prove this by induction.
Consider two players A and B. If A defeats B, we define $P_1 = A$ and $P_2 = B$. Otherwise, $P_1 = B$ and $P_2 = A$.
Assume the statement for some $k > 2$ players.
Now, we prove the statement for $k+1$ players. Take any k of these players. By the induction hypothesis, they can be arranged as $P_1, P_2, ..., P_k$ satisfying the given condition. Let the remaining player be Q. Let i be the least index such that Q defeats P_i. If $i = 1$, place Q before P_1 and relabel. Otherwise, check if Q is defeated by P_{i-1}. If so, then place Q between P_{i-1} and P_i and relabel. If not, look at the next i such that Q defeats P_i and check the same. If no such i exists, then P_k defeats Q and we can place Q at the end and relabel.

□

Q2

Consider the polynomial $ax^3 + bx^2 + cx + d$, where ad is odd and bc is even. Prove that all roots of the polynomial cannot be rational.

Solution

Let the roots α_1, α_2, α_3 be rational. We write them as $\alpha_i = \frac{p_i}{q_i}$ (in least terms), $i = 1, 2, 3$. Then,

$$a\alpha_i^3 + b\alpha_i^2 + c\alpha_i + d = 0$$
$$\Rightarrow ap_i^3 + bp_i^2 q_i + cp_i q_i^2 + dq_i^3 = 0$$
$$\Rightarrow ap_i^3 = -q_i(bp_i^2 + cp_i q_i + dq_i^2)$$

$\Rightarrow q_i | ap_i^3$. But, $\gcd(p_i, q_i) = 1$. Hence, $q_i | a \Rightarrow q_i | ad \Rightarrow q_i$ is odd, since ad is odd. Similarly, $p_i | d \Rightarrow p_i | ad \Rightarrow p_i$ is odd. Now, $\alpha_1 + \alpha_2 + \alpha_3 = -\frac{b}{a}$. Hence,

$$b = -\frac{a}{q_1 q_2 q_3}(p_1 q_2 q_3 + p_2 q_1 q_3 + p_3 q_1 q_2)$$

But, p_i and q_i are all odd $\Rightarrow p_1 q_2 q_3 + p_2 q_1 q_3 + p_3 q_1 q_2$ is odd. Also, ad is odd $\Rightarrow a$ is odd $\Rightarrow b$ is odd.
Similarly,

$$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \frac{c}{a}$$
$$\Rightarrow c = \frac{a}{q_1 q_2 q_3}(p_1 p_2 q_3 + p_2 p_3 q_1 + p_3 p_1 q_2)$$

Then, c is odd $\Rightarrow bc$ is odd. Contradiction.

□

Q3

$P(x) = x^n + a_1 x^{n-1} + ... + a_n$ is a polynomial with real coefficients. $a_1^2 < a_2$. Prove that all roots of $P(x)$ cannot be real.

Solution

Suppose P(x) has all n roots real then $P^{(1)}(x)$ has only (n-1) real roots. [By Rolle's Theorem $P^{(1)}(x)$ has a real root between two roots of P(x)]
Then the $(n-2)^{th}$ derivative of $P(x)$, $P^{(n-2)}(x)$, has two real roots.

$$P^{(n-2)}(x) = \frac{n!}{2}x^2 + (n-1)!a_1 x + (n-2)!a_2$$

Then the discriminant,

$$((n-1)!a_1)^2 - \frac{4n!}{2}(n-2)!a_2 = (n-1)!(n-2)!((n-1)a_1^2 - 2na_2)$$

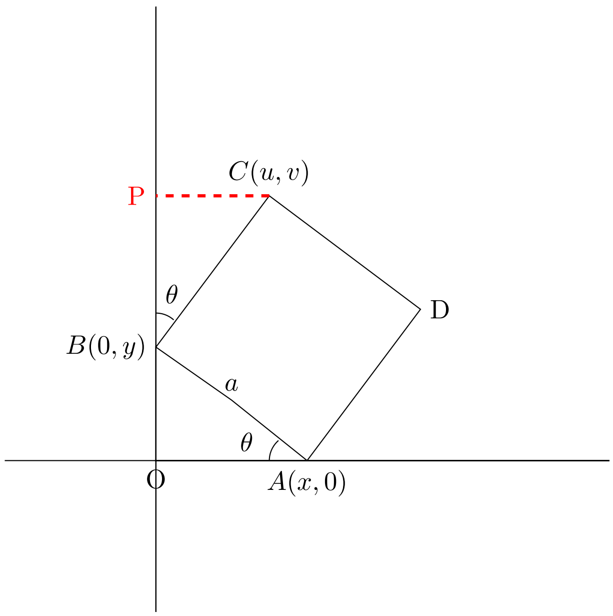
is positive. So, we have $a_1^2 < a_2 \Rightarrow (n-1)a_1^2 < (n-1)a_2 < 2na_2$
This implies that the discriminant of $P^{(n-2)}(x)$ is negative . Contradiction.

□

Q4

Let $ABCD$ be a square. Let A lie on the positive x-axis and B on the positive y-axis. Suppose the vertex C lies in the first quadrant and has co-ordinates (u, v). Then find the area of the square in terms of u and v.

Solution



Draw a perpendicular line CP to the y-axis.
Let $\angle OAB = \theta \Rightarrow \angle PBC = \theta$
Let $AB = BC = a \Rightarrow BP = a\cos\theta, PC = a\sin\theta$
Let $A \equiv (x, 0), B \equiv (0, y)$
So, $OB = a\sin\theta = y$, $OA = a\cos\theta = x$
$u = CP = a\sin\theta = y$, $v = OP = OB + BP = y + a\cos\theta = y + x$
Hence the area of the square is $a^2 = x^2 + y^2 = u^2 + (v - u)^2$.

Q5

Prove that there exists a right-angled triangle with rational sides and area d iff there exist rational numbers x, y, z such that x^2, y^2, z^2 are in arithmetic progression with common difference d.

Solution

(⇒)
Let △ be a right-angled triangle with rational sides a, b, c and $a^2 + b^2 = c^2$.
Area of △ is $d = \frac{ab}{2}$
Let $x = \frac{a-b}{2}$, $y = \frac{c}{2}$, $z = \frac{a+b}{2}$
The,

$$y^2 - x^2 = \frac{c^2}{4} - \frac{(a-b)^2}{4} = \frac{c^2 - a^2 - b^2 + 2ab}{4} = \frac{ab}{2} = d$$

$$\Rightarrow z^2 - y^2 = \frac{(a+b)^2}{4} - \frac{c^2}{4} = \frac{a^2 + b^2 + 2ab - c^2}{4} = \frac{ab}{2} = d$$

Hence x^2, y^2, z^2 are in A.P. with common difference d.

(⇐)
x^2, y^2, z^2 are in A.P. with common difference d.
So, $d = y^2 - x^2 = z^2 - y^2$ and $x^2 + z^2 = 2y^2$
. Let $a = x + z$, $b = z - x$, $c = 2y$
Then,

$$a^2 + b^2 = (z+x)^2 + (z-x)^2 = 2(x^2 + z^2) = 4y^2 = c^2$$

This implies that a,b,c are sides of a right-angled triangle.
So, the area of the triangle is

$$\frac{ab}{2} = \frac{(z+x)(z-x)}{2} = \frac{z^2 - x^2}{2} = \frac{z^2 - y^2 + y^2 - x^2}{2} = \frac{2d}{2} = d$$

.

□

Q6

Suppose in a $\triangle ABC$, A, B, C denote the three angles and a, b, c denote the three sides opposite to the corresponding angles. Prove that, if $sin\ (A - B) = \frac{a}{a+b} \sin A \cos B - \frac{b}{a+b} \sin B \cos A$, then $\triangle ABC$ is isosceles.

Solution

$$\sin(A - B) = \frac{a}{a+b} \sin A \cos B - \frac{b}{a+b} \sin B \cos A$$

$$\Rightarrow \sin A \cos B - \sin B \cos A = \frac{a}{a+b} \sin A \cos B - \frac{b}{a+b} \sin B \cos A$$

$$\Rightarrow \left(1 - \frac{a}{a+b}\right) \sin A \cos B = \left(1 - \frac{b}{a+b}\right) \sin B \cos A$$

$$\Rightarrow \frac{a}{a+b} \sin A \cos B = \frac{b}{a+b} \sin B \cos A$$

$$\Rightarrow \left(\frac{b}{sinB}\right)\cos B = \left(\frac{a}{\sin A}\right)\cos A$$

We know

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

Hence, $\cos B = \cos A \Rightarrow$ either $A = B$ or $A + B = 2\pi$ The latter is not possible as $A + B < \pi$. Hence $A = B$. So $\triangle ABC$ is isosceles.

□

Q7

f is a differentiable function, such that $f(f(x)) = x$, where $x \in [0, 1]$. Also, $f(0) = 1$. Find the value of

$$\int_0^1 (x - f(x))^{2016} dx$$

Solution

Let

$$I = \int_0^1 (x - f(x))^{2016} dx$$

Let $y = f(x),\ f(f(x)) = x \Rightarrow f'(f(x))f'(x) = 1 \Rightarrow f'(x) = \frac{1}{f'(y)}$ Then, $dy = f'(x)dx \Rightarrow f'(y)dy = dx$ and $f(0) = 1,\ f(f(0)) = 0 \Rightarrow f(1) = 0$
Now,

$$I = \int_0^1 (x - f(x))^{2016} dx = \int_1^0 (f(y) - y)^{2016} f'(y) dy$$

$$= \int_1^0 (f(y) - y)^{2016} (f'(y) - 1) dy - I$$

$$2I = \int_1^0 (f(y) - y)^{2016} (f'(y) - 1) dy$$

let $z = f(y) - y, \Rightarrow dz = (f'(y) - 1)dy \begin{cases} y = 0, z = 1 \\ y = 1, z = -1 \end{cases}$

$$2I = \int_{-1}^1 z^{2016} dz$$

$I = \frac{1}{2 \cdot 2017} \left[z^{2017}\right]_{-1}^1 = \frac{2}{2 \cdot 2017} = \frac{1}{2017}$.

Q8

$(a_n)_{n\geq 1}$ is a sequence of real numbers satisfying $a_{n+1} = \frac{3a_n}{2+a_n}$.
(i) If $0 < a_1 < 1$, then prove that the sequence a_n is increasing and hence,

$$\lim_{n\to\infty} a_n = 1$$

.
(ii) If $a_1 > 1$, then prove that the sequence a_n is decreasing and hence,

$$\lim_{n\to\infty} a_n = 1$$

.

Solution

(i). We have $0 < a_1 < 1$ and $a_{n+1} = \frac{3a_n}{2+a_n} \ldots (*)$

$$\frac{a_{n+1}}{a_n} = \frac{3}{2+a_n}$$

$0 < a_1 < 1 \Rightarrow 2 + a_1 < 3$
For $n = 1$, $\frac{a_2}{a_1} = \frac{3}{2+a_1} > 1 \Rightarrow a_2 > a_1 > 0$.
Now $a_2 = \frac{3a_1}{2+a_1} \Rightarrow a_1 = \frac{2a_2}{3-a_2}$
$0 < a_1 < 1 \Rightarrow 0 < \frac{2a_2}{3-a_2} < 1 \ldots (**)$
Now consider $0 < \frac{2a_2}{3-a_2} \Rightarrow 3 - a_2 > 0$ since $a_2 > 0$
From (**) we get

$$\frac{2a_2}{3-a_2} < 1 \Rightarrow 2a_2 < 3 - a_2 \Rightarrow 3a_2 < 3 \Rightarrow a_2 < 1$$

So, we get $0 < a_2 < 1$
By similar argument we can show $a_3 > a_2 > 0$ and $0 < a_3 < 1$ and so on.
So, we get $(a_n)_{n\geq 1}$, an increasing sequence and $0 < a_n < 1$ $\forall n \geq 1$ so bounded. $\Rightarrow (a_n)_{n\geq 1}$ has a limit. Let

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

Taking limit of both sides of (*) we get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3a_n}{2+a_n}$$

$$\Rightarrow a = \frac{3a}{2+a}$$

$$\Rightarrow a^2 - a = 0$$

$$\Rightarrow a = 1 \ \ or \ \ 0$$

$(a_n)_{n\geq 1}$, an increasing sequence and $a_n > 0$
hence a$\neq 0 \Rightarrow \lim_{n\to\infty} a_n = a = 1$.

□

(ii). We have

$$a_1 > 1 \Rightarrow 2 + a_1 > 3 \Rightarrow 1 > \frac{3}{2+a_1} = \frac{a_2}{a_1} \Rightarrow a_2 < a_1$$

since $a_1 > 1 \Rightarrow 1 > \frac{1}{a_1}$

$$\Rightarrow 3 > \frac{2}{a_1} + 1 \Rightarrow \frac{3}{\frac{2}{a_1}+1} > 1$$

Hence

$$a_2 = \frac{3a_1}{2+a_1} = \frac{3}{\frac{2}{a_1}+1} > 1$$

Similarly we can show $a_n > 1 \ \forall \ n \geq 1$ and $(a_n)_{n\geq 1}$, a decreasing sequence having lower bound 1 .
$\Rightarrow (a_n)_{n\geq 1}$ has a limit .
let

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

Taking limit of both sides of (*) we get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3a_n}{2+a_n}$$

$$\Rightarrow a = \frac{3a}{2+a}$$

$$\Rightarrow a^2 - a = 0$$

$$\Rightarrow a = 1 \ \ or \ \ 0$$

Since $a_n > 1 \ \forall \ n \geq 1$, $a \neq 0 \Rightarrow \lim_{n\to\infty} a_n = a = 1$.

□